

## Time reversal and exceptional points

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**Abstract.** Eigenvectors of decaying quantum systems are studied at exceptional points of the Hamiltonian. Special attention is paid to the properties of the system under time reversal symmetry breaking. At the exceptional point the chiral character of the system — found for time reversal symmetry — generically persists. It is, however, no longer circular but rather elliptic.

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In a system described by a non-Hermitian Hamiltonian  $H$ , a surprising phenomenon can occur: the coalescence of two eigenmodes. This means that two eigenvalues merge such that there is only one eigenvector. As a consequence,  $H$  cannot be diagonalized by a similarity transformation. Considering  $H$  to depend on some parameter  $\lambda$ , the value  $\lambda_c$ , where this happens, is called an exceptional point (EP) [1]. It is well-known that Hermitian operators cannot have any EPs: at a degeneracy of two of their eigenvalues, the space of eigenvectors is two-dimensional.

In the vicinity of an EP, the eigenvalues *and* eigenvectors show branch point singularities [2–5] as functions of  $\lambda$ . This contrasts with a two-fold degeneracy in a hermitian matrix, where no singularity but rather a diabolic point [6] occurs. EPs have been observed in laser induced ionization of atoms [7], in acoustical systems [8], microwave cavities [9,10], in optical properties of certain absorptive media [11–13], and in “crystals of light” [14]. Models for Stark resonances in atomic physics have been analysed in terms of EPs and their connection to diabolic points discussed [15]. The broad variety of physical systems showing EPs indicates that their occurrence is generic.

The observation of EPs is possible in decaying quantum systems. So far only complex symmetric Hamiltonians have been considered. Such “effective” Hamiltonians are used to model decaying or resonant systems when invariance under time reversal prevails. They are obtained by eliminating open decay channels from explicit consideration. Therefore, the possibility for the system to decay is not at variance with time reversal symmetry.

For complex symmetric  $H$ , a recent theoretical paper [16] has found the eigenfunction  $|\psi_{\text{EP}}\rangle$  at the EP to

be of the form

$$|\psi_{\text{EP}}\rangle = \begin{pmatrix} \pm i \\ 1 \end{pmatrix}. \quad (1)$$

The phase difference  $\pm i$  between the components of the state vector is independent of an arbitrary two-dimensional orthogonal transformation [16]; in fact, the  $|\psi_{\text{EP}}\rangle$  are eigenstates of an orthogonal transformation and are therefore independent of a particular choice of the basis. The phase  $i$  has been confirmed experimentally [17].

In the present note, we again address the eigenfunction at the EP — for a situation, where time reversal symmetry is broken. This does not necessarily mean that it is broken on a fundamental level as it is in the system of the neutral  $K$ -mesons. An external magnetic field applied to a moving charge provides time reversal symmetry breaking. Similarly, one can introduce magnetic elements into microwave cavities that allow only one direction for a traveling wave [18].

In the vicinity of an EP, where two (and only two) eigenvalues merge, an  $n$ -dimensional system can locally be represented by a two-state system [16]. Therefore we confine ourselves to two-dimensional  $H$  in the sequel.

Let  $H$  be the sum

$$H = H_0 + \lambda H_1 \quad (2)$$

of two Hermitian operators  $H_0, H_1$  with  $H_1$  multiplied by a complex strength parameter  $\lambda$ . In order that there be exceptional points  $\lambda_c$ , the operators  $H_0$  and  $H_1$  must not commute. If they do,  $H$  can be diagonalized for every complex  $\lambda$ . We write

$$H_0 = U_0 \epsilon U_0^\dagger \quad \text{and} \quad H_1 = U_1 \omega U_1^\dagger, \quad (3)$$

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where

$$\epsilon = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}, \quad (4)$$

and  $U_0, U_1$  are unitary matrices. Throughout the present paper we assume that  $\epsilon$  and  $\omega$  are different from a multiple of the unit matrix. Then  $H_0, H_1$  do not commute if  $U_0, U_1$  are different from each other.

The  $U_k$  can be parameterized by two angles  $\phi$  and  $\tau$  such that

$$U_k = U(\phi_k, \tau_k) \quad \text{where} \quad k = 0, 1 \quad (5)$$

and

$$U(\phi, \tau) = \begin{pmatrix} \cos \phi & -\sin \phi \exp(i\tau) \\ \sin \phi \exp(-i\tau) & \cos \phi \end{pmatrix}. \quad (6)$$

A general unitary transformation  $U_g$  in two dimensions actually has four parameters. It can be represented as

$$U_g = U \begin{pmatrix} \exp(i\gamma_1) & 0 \\ 0 & \exp(i\gamma_2) \end{pmatrix}. \quad (7)$$

Since  $H_k$  is independent of the phases  $\gamma_1, \gamma_2$ , we may set them equal to zero. Thus, equation (6) is the most general unitary transformation in the context of equation (3).

The complex strength parameter  $\lambda$  allows for the system to decay. Since we assume that the time reversal operator  $\mathcal{T}$  equals the complex conjugation  $K$ , the Hamiltonian  $H$  is time reversal symmetry breaking if  $\tau_0 \neq 0$  or  $\tau_1 \neq 0$ .

Let us discuss the consequences of this model in two steps: first, the simplification  $\tau_1 = \tau_2$  is introduced, and second, the general case is discussed.

*A special case of time reversal symmetry breaking.* We assume that  $\tau_0$  and  $\tau_1$  equal each other, i.e.

$$\tau = \tau_0 = \tau_1. \quad (8)$$

At the EPs, the eigenvectors are

$$|\psi_{\text{EP}}^{\pm}\rangle \propto \begin{pmatrix} \pm i e^{i\tau} \\ 1 \end{pmatrix}, \quad (9)$$

respectively. One recognizes this by using the time reversal symmetric result (1) together with the observation that the present  $H$  can be written in the form

$$H = z(\tau) (U(\phi_0, 0)\epsilon U^{\dagger}(\phi_0, 0) + \lambda U(\phi_1, 0)\omega U^{\dagger}(\phi_1, 0)) z^{\dagger}(\tau), \quad (10)$$

where  $z$  is the matrix

$$z(\tau) = \begin{pmatrix} \exp(i\tau/2) & 0 \\ 0 & \exp(-i\tau/2) \end{pmatrix}. \quad (11)$$

The result (9) differs from equation (10) of [16] by the phase  $\tau$ . For  $\tau = 0$ , one retrieves the time reversal invariant situation described there.

The left hand eigenvectors  $\langle \tilde{\psi}_{\text{EP}}^{\pm} |$  at the EPs are

$$\langle \tilde{\psi}_{\text{EP}}^{\pm} | = (\pm i e^{-i\tau}, 1). \quad (12)$$

This differs from equation (5) of [16] in that  $|\tilde{\psi}_{\text{EP}}^{\pm}\rangle$  is not the complex conjugate of the right hand eigenvector  $|\psi_{\text{EP}}^{\pm}\rangle$ . Note, however, that relation (9) of [16] persists: the inner product of the left and right hand eigenvectors vanishes, viz.

$$\langle \tilde{\psi}_{\text{EP}}^{\pm} | \psi_{\text{EP}}^{\pm} \rangle = 0. \quad (13)$$

Therefore the biorthogonal normalization is impossible at the EP.

The present case covers the even more special situation where  $\phi_0 = 0$ , whence

$$H_0 = \epsilon. \quad (14)$$

The evaluation of this case provides the basis for the later evaluation of the general case. If (14) holds, the eigenvalues are

$$E_{1,2} = \frac{1}{2} \left( \epsilon_1 + \epsilon_2 + \lambda(\omega_1 + \omega_2) \right) \pm R, \quad (15)$$

where

$$R = \frac{1}{2} \left\{ (\epsilon_1 - \epsilon_2)^2 + \lambda^2(\omega_1 - \omega_2)^2 + 2\lambda(\epsilon_1 - \epsilon_2)(\omega_1 - \omega_2) \cos 2\phi_1 \right\}^{-\frac{1}{2}}. \quad (16)$$

The two levels coalesce at  $R = 0$ . This yields two EPs at

$$\lambda_c^{\pm} = -\frac{\epsilon_1 - \epsilon_2}{\omega_1 - \omega_2} e^{\pm 2i\phi_1}. \quad (17)$$

Note that by our assumptions  $\epsilon_1 \neq \epsilon_2$ ,  $\omega_1 \neq \omega_2$  and  $\phi_1 \neq 0$ .

For a given  $\tau$ , the transformations  $U(\phi, \tau)$  with  $-\pi \leq \phi < \pi$  form a subgroup of the unitary matrices. The eigenvectors (9) are invariant under the transformations of this group because  $|\psi_{\text{EP}}^{\pm}\rangle$  is eigenvector of every element of the group, i.e.

$$U(\phi, \tau) |\psi_{\text{EP}}^{\pm}\rangle = \exp(\pm i\phi) |\psi_{\text{EP}}^{\pm}\rangle. \quad (18)$$

This parallels the result stated above and in equation [16] in that the vector given in equation (1) is an eigenvector of all orthogonal transformations. In the more general cases of time reversal symmetry breaking there is no longer such symmetry.

*The general case of time reversal symmetry breaking.*

We assume that both,  $\phi_0$  and  $\phi_1$ , are different from zero and  $\tau_0 \neq \tau_1$ .

One can, of course, transform to the eigenbasis of  $H_0$ . Then  $H$  takes the diagonal form used above. However,  $\mathcal{T}$  is no longer equal to  $K$ , it rather is  $\mathcal{T} = U^{\dagger}(\phi_0, \tau_0) K U(\phi_0, \tau_0)$ . By this change of the basis, we obtain the general form of the eigenfunction at the EP.

The diagonalisation of  $H_0$  brings  $H$  into the form

$$\begin{aligned} \tilde{H} &= \epsilon + \lambda U^{\dagger}(\phi_0, \tau_0) H_1 U(\phi_0, \tau_0) \\ &= \epsilon + \lambda U^{\dagger}(\phi_0, \tau_0) U(\phi_1, \tau_1) \omega U^{\dagger}(\phi_1, \tau_1) U(\phi_0, \tau_0). \end{aligned} \quad (19)$$

According to the above discussion, the eigenfunction at the EP is given by

$$|\psi_{\text{EP}}^{\pm}\rangle_{\tilde{H}} \propto \begin{pmatrix} \pm i e^{i\xi} \\ 1 \end{pmatrix}, \quad (20)$$

where the phase  $\xi$  and the position of the EP are functions of  $\phi_0, \tau_0, \phi_1, \tau_1$ . The explicit dependence is deferred to the appendix. As a result, the state vector of the Hamiltonian  $H$  at the EP is given by

$$|\psi_{\text{EP}}^{\pm}\rangle_H \propto U(\phi_0, \tau_0) \begin{pmatrix} \pm i e^{i\xi} \\ 1 \end{pmatrix} \quad (21)$$

which may be rewritten as

$$|\psi_{\text{EP}}^{\pm}\rangle_H \propto \begin{pmatrix} \pm i e^{-i\xi} (e^{2i\xi} \cos^2 \phi_0 + e^{2i\tau_0} \sin^2 \phi_0) \\ 1 \pm \sin(2\phi_0) \sin(\tau_0 - \xi) \end{pmatrix}. \quad (22)$$

Here, the lower component has been chosen real. Note that the scalar product of the left hand and right hand eigenvector again vanishes as in equation (13).

We discuss this result. (i) For  $\phi_0 = 0$ , corresponding to diagonal  $H_0$ , the form of equation (9) is retrieved from (22). (ii) For  $\tau_0 = 0$ , corresponding to time reversal invariant  $H_0$ , the upper component of (22) simplifies to  $\pm i \cos(\xi) \mp \cos(2\phi_0) \sin \xi$ . Even in this case, the ratio between the upper and lower component is not a phase factor but can assume any value in the complex plane. Of course, large values of the ratio correspond to small values of the lower component which can in fact vanish. (iii) If  $\tau_0 = \tau_1$  while maintaining  $\phi_0 \neq \phi_1$ , equation (22) leads back to equation (9).

*In conclusion:* the ‘‘universal’’ phase of  $\pi/2$  in the eigenvector at an EP is no longer universal if time reversal symmetry is broken. The phase as well as the relative amplitude in the eigenvector at the EP can be manipulated in an experiment that allows to control the violation of this symmetry.

If — in analogy with wave optics [13] and as was done in [17] — one associates a circularly polarized wave with the eigenvector (1), time reversal symmetry breaking leads to the elliptically polarized wave (22). An elliptical wave can be generated in two ways: the phase of the upper component in (22) is different from  $\pm\pi/2$  or the amplitude ratio is not unity. The limit of a linearly polarized wave is obtained when the upper component is real. If this happens, the chiral character of the wave function at an EP is lost. We emphasize, however, that even this dramatic effect of time reversal symmetry breaking upon the wave function does not alter the fact that only one mode can occur at the EP. Time reversal symmetry breaking simply changes the amplitude ratio such that it may assume any complex value. This includes in particular the special cases where either the upper or the lower component in (22) vanishes.

We believe that these results can be verified in future experiments with quantum dots in a magnetic field and with microwave cavities containing suitable magnetic elements [18].

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## Appendix A: The phase $\xi$

The product

$$U_0^\dagger U_1 = U^\dagger(\phi_0, \tau_0) U(\phi_1, \tau_1) \quad (A.1)$$

is equal to

$$U_0^\dagger U_1 = \begin{pmatrix} \cos \phi_0 \cos \phi_1 & \cos \phi_1 \sin \phi_0 e^{i\tau_0} \\ + \sin \phi_0 \sin \phi_1 e^{i(\tau_0 - \tau_1)} & - \cos \phi_0 \sin \phi_1 e^{i\tau_1} \\ - \cos \phi_1 \sin \phi_0 e^{-i\tau_0} & \cos \phi_0 \cos \phi_1 \\ + \cos \phi_0 \sin \phi_1 e^{-i\tau_1} & + \sin \phi_0 \sin \phi_1 e^{-i(\tau_0 - \tau_1)} \end{pmatrix}. \quad (A.2)$$

We introduce the phase of the (1,1)-element, viz.

$$\gamma = \arg \left( \cos \phi_0 \cos \phi_1 + \sin \phi_0 \sin \phi_1 e^{i(\tau_0 - \tau_1)} \right), \quad (A.3)$$

and write equation (A.1) in the form

$$U_0^\dagger U_1 = \begin{pmatrix} \cos \beta & -\sin \beta e^{i\xi} \\ \sin \beta e^{-i\xi} & \cos \beta \end{pmatrix} z(2\gamma). \quad (A.4)$$

Here,  $z$  is the matrix defined in equation (11), and  $\beta$  is given by

$$\cos \beta = \left( \cos^2 \phi_0 \cos^2 \phi_1 + \sin^2 \phi_0 \sin^2 \phi_1 + 2 \cos \phi_0 \cos \phi_1 \sin \phi_0 \sin \phi_1 \cos(\tau_0 - \tau_1) \right)^{1/2} \quad (A.5)$$

and  $\xi$  by

$$\xi = \arg \left( \cos \phi_1 \sin \phi_0 e^{i\tau_0} - \cos \phi_0 \sin \phi_1 e^{i\tau_1} \right) + \gamma. \quad (A.6)$$

The exceptional point occurs at

$$\lambda_{\text{EP}} = -\frac{\epsilon_1 - \epsilon_2}{\omega_1 - \omega_2} e^{\pm 2i\beta}. \quad (A.7)$$

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